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An Algorithm for the Orientation of Complete Bipartite Graphs

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***Abstract*—Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We consider the problem of orienting the edges of a complete bipartite graph $K_{n,n}$ so only two different in-degrees a and b occur. An obvious necessary condition for orienting the edges of G so that only two in-degrees a and b occur, is that there exist positive integers s and t satisfying $s+t=|V(G)|$ and $as+bt=|E(G)|$. In this paper, we show that the necessary condition is also sufficient for a complete bipartite graph $K_{n,n}$. Furthermore, we give the algorithms of orientations with only two in-degrees of $K_{n,n}$.**

Keywords-complete bipartite graph; orientation; algorithm

I. INTRODUCTION

An orientation $D=(V(D),A(D))$ of an undirected graph $G=(V(G),E(G))$ is a digraph obtained by replacing each undirected edge $e\in E$ with an arc from one end vertex of e to the other. In the oriented problem, we are asked whether G has an orientation satisfying some conditions. This is a basic problem in combinatorial optimization, and many beautiful results have been produced so far. Chen et al. [3] studied orientations of graphs satisfying the Ore condition. Fukunaga [4] investigated graph orientations with set connectivity requirements. Miao and Lin [7] gave strong orientations of complete k-partite graphs achieving the strong diameter. The main purpose of this paper is to orient $K_{n,n}$ with a or b arrowheads directed towards each vertex.

Usually a digraph has many different in-degrees. This paper is to orient graph $K_{n,n}$ achieving only two in-degrees. This kind of oriented problem is useful in practice. Buhler et al. [1] considered the problem of orienting the edges of the n-dimensional hypercube so that only two in-degrees occur for finding strategies for specific hat guessing games.

Let D be a digraph, for any $uv\in A(D)$, we say that u dominates v (or v is dominated by u) and denote it by $u\to v$. For any $v\in V(D)$, the in-degree of v is denoted by $d^-(v)=|\{u\in V(D): uv\in A(D)\}|$ and the out-degree of v is denoted by $d^+(v)=|\{u\in V(D): vu\in A(D)\}|$. For disjoint subsets X and Y of $V(D)$, $X\to Y$ means that every vertex of X dominates every vertex of Y, and we define $[X,Y]=\{xy\in A(D): x\in X, y\in Y\}$. For graph-theoretical terminology and notation not defined here we follow[2, 6].

In this paper, $K_{n,n}$ is oriented to a digraph so that only two in-degrees a and b occur. For convenience, let $[a,b]_n$ be a shorthand for the problem of realizing an orientation of $K_{n,n}$ whose only in-degrees are a or b. In Section 2, we give a necessary and sufficient condition such that $[a,b]_n$ is realizable. In Section 3, we give some specified algorithms to construct the required orientations of $K_{n,n}$.

II. MAIN RESULTS

Lemma 2.1. Given a positive integer n, let $K_{n,n}$ be a complete bipartite graph. For $a,b\in\{0,1,2,\ldots,n\}$, if $[a,b]_n$ is realizable, then there exist positive integers s and t satisfying the following two equations:

$$\begin{cases} s+t=2n, \\ as+bt=n^2. \end{cases}$$

Proof. Let $[a,b]_n$ be realizable. Then there exixts an oriented graph whose only in-degrees are a or b. Let the in-degree of s vertices in $K_{n,n}$ be a. Then the in-degree of the remaining $(2n-s)$ vertices is b. Therefore, $s+t=|V(K_{n,n})|=2n$, and $as+bt=|E(K_{n,n})|=n^2$, where $t=n-s$.

Let G be a nonoriented graph. For $U\subseteq V(G)$, denote the number of edges which have their both end-vertices in U by $e(U)$.

Lemma 2.2. [5] Given a nonoriented graph G whose vertices are labeled $v_1, v_2, \cdots, v_n$ and to whose vertices are associated non-negative integers $\nu(v_1), \nu(v_2), \cdots, \nu(v_n)$, respectively; then, G is orientable with $\nu(v_i)$ arrowheads directed toward vertex v_i (for $i \in \{1, 2, \ldots, n\}$) if and only if $\sum_{v \in V(G)} \nu(v) = |E(G)|$ and

$$e(U) \le \sum_{v \in U} \nu(v) \quad \text{for each } U \subseteq V(G). \tag{1}$$

Lemma 2.3. Given three positive integers n, s and t, let $K_{n,n}$ be a complete bipartite graph and let $a, b \in \{0, 1, 2, \ldots, n\}$ with $a \le b$. If a, b, s, t satisfy the following two equations:

$$\begin{cases} s + t = 2n, & (2) \\ as + bt = n^2. & (3) \end{cases}$$

Then $[a, b]_n$ is realizable.

Proof. Case 1. $s > t$.

By $a \le b$ and $a, b \in \{0, 1, 2, \ldots, n\}$, $n - a \ge n - b$ and $n - a, n - b \in \{0, 1, 2, \ldots, n\}$. We can deduce that $(n-a)s + (n-b)t = (n-a)s + (2n-s)(n-b) = 2n^2 - [as + b(2n-s)]$ $= 2n^2 - n^2 = n^2$. Set $a' = n - b$, $b' = n - a$, $s' = t$, $t' = s$. Now, $a' \le b'$ and $s' < t'$. Then \$a', b', s', t'\$ satisfy the conditions of Case 1. By the proof of Case 1, $[a', b']_n$ is realizable. Then $K_{n,n}$ has an orientation D whose only in-degrees are a' or b'. We consider the digraph D' obtained by reversing all the arcs in D. Note that $K_{n,n}$ is n-regular. Then $[n - a', n - b']_n$ is realizable, i.e. $[b, a]_n$ is realizable. So $[a, b]_n$ is realizable.

Case 2. $s \le t$.

The proof of this case is similar to Case 1.

Lemma 2.4. Given three positive integers n, s and t, let $K_{n,n}$ be a complete bipartite graph and let $a, b \in \{0, 1, 2, \ldots, n\}$ with $a > b$. If a, b, s, t satisfy the following two equations:

$$\begin{cases} s + t = 2n, \\ as + bt = n^2. \end{cases}$$

Then $[a, b]_n$ is realizable.

Proof. By $a > b$ and $a, b \in \{0, 1, 2, \ldots, n\}$, $n - a < n - b$ and $n - a, n - b \in \{0, 1, 2, \ldots, n\}$. We can deduce that $(n-a)s + (n-b)t = (n-a)s + (2n-s)(n-b) = 2n^2 - [as + b(2n-s)]$ $= 2n^2 - n^2 = n^2$. Then $n - a$ and $n - b$ satisfy the conditions of Lemma 2.3. By Lemma 2.3, $[n-a, n-b]_n$ is realizable. Then $K_{n,n}$ has an orientation D whose only in-degrees are $n - a$ or $n - b$. We consider the digraph D' obtained by reversing all the arcs in D. Note that $K_{n,n}$ is n-regular. Then $[n - (n-a), n - (n-b)]_n = [a, b]_n$ is realizable.

By Lemmas 2.1, 2.3 and 2.4, we can obtain the following theorem directly.

Theorem 2.5. Given a positive integer n, let $K_{n,n}$ be a complete bipartite graph. For $a, b \in \{0, 1, 2, \ldots, n\}$, $[a, b]_n$ is realizable if and only if there exist positive integers s and t satisfying the following two equations:

$$\begin{cases} s + t = 2n, \\ as + bt = n^2. \end{cases}$$

Corollary 2.6. Given a positive integer n, let $K_{n,n}$ be a complete bipartite graph. For $a, b \in \{0, 1, 2, \ldots, n\}$, the following results hold:

(a) if $[a, b]_n$ is realizable, then $[n-a, n-b]_n$ is realizable;

(b) $[0, n]_n$ is realizable;

(c) if n is even, then $[\frac{n}{2}, \frac{n}{2}]_n$ is realizable.

III. Orientation Algorithms of $K_{n,n}$

In Section 2, we have proved that $K_{n,n}$ admits the orientation with only two in-degrees. In this section, we will show how to orient $K_{n,n}$ by specified algorithms. By the proofs of Lemmas 2.3 and 2.4, it is enough to consider the case where $a \le b$ and $s \le t$.

Specially, $K_{1,1}$ has an orientation D whose only in-degrees are 0 or 1. Then $[0,1]_1$ is obviously realizable. In the following, suppose that $n \ge 2$.

If $a = b$, then, by the equations (2) and (3), $a = b = \frac{n}{2}$. Note that a is an integer and $a = \frac{n}{2}$. Therefore, n is even. Combining this with the fact that the degree of each vertex in $K_{n,n}$ is n, $K_{n,n}$ admits an Euler tour. By the definition of Euler tour, there exists an oriented graph of $K_{n,n}$ whose only in-degree is $\frac{n}{2}$. Then $[\frac{n}{2}, \frac{n}{2}]_n$ is realizable. Next assume that $a < b$. By the equations (2) and (3), we have $s = \frac{n(2b-n)}{b-a}$

and $t=\dfrac{n(n-2a)}{b-a}$. Combining this with the fact that s and t are positive integers, $a<\dfrac{n}{2}<b$.

By $b>\dfrac{n}{2}$, $b>n-b$. Since $s\le t$ and $s+t=2n$, $n-s\ge 0$. We can deduce that $n^2=as+bt=as+b(2n-s)=as+bn+b(n-s)\ge$ $as+bn+(n-b)(n-s)=as+bn+n^2-bn-sn+bs=(a+b-n)s+n^2$ Hence $(a+b-n)s\le 0$. Combining this with $s>0$, $a+b\le n$.

Case 1. $a+b=n$.

Let (X,Y) be a bipartition of $K_{n,n}$ with $X=\{u_0,u_1,\dots,u_{n-1}\}$ and $Y=\{v_0,v_1,\dots,v_{n-1}\}$. By $a+b=n$, $s=\dfrac{n(2b-n)}{b-a}=\dfrac{n(2b-n)}{b-(n-b)}=\dfrac{n(2b-n)}{2b-n}=n$, i.e., $s=n$. Combining this with $s+t=2n$, $t=n$. Then $s=t=n$. Conversely, if $s=t$, then By the equations (2) and (3), we have $s=t$ and $a+b=n$. If $a=0$, then $b=n$. Construct a special orientation such that $X\to Y$. Then the in-degree of each vertex in X is 0 and the in-degree of each vertex in Y is n. Therefore, $[0,n]_n$ is realizable. Next, assume that $a>0$. For any $u_i\in X$, orient $v_{(i+l)(\bmod n)}\to u_i$ for each $l=0,1,\dots,a-1$. We orient the remaining edges which are incident to u_i towards Y. Now, we obtain an oriented graph D. The in-degree of each vertex of X in D is a. For any $v_i\in Y$, by the definition of D, $v_i\to u_{(i-l)(\bmod n)}$ for each $l=0,1,\dots,a-1$. The out-degree of each vertex of Y in D is a, and the in-degree of each vertex of Y in D is $n-a$. Only two in-degrees a and $n-a$ occur in D. Set $b=n-a$. Therefore, $[a,b]_n$ is realizable. Then we can obtain the following proposition directly.

Proposition 3.1. Let $a<b$ and $a+b=n$. Then the oriented graph which is obtained by the above method has only two in-degrees a and b.

Case 2. $a+b<n$.

In this case, $s<t$. By $a+b<n$ and $a<b$, $n-s=n-\dfrac{n(2b-n)}{b-a}=\dfrac{(b-a)n-n(2b-n)}{b-a}=\dfrac{n(n-a-b)}{b-a}>0$, i.e., $s<n$. Combining this with $s+t=2n$, $t>n$. Then $s<n<t$.

First, assume that $a=0$. By $a+b<n$, $b<n$. If $s\ge b$, $n^2=as+bt=as+b(2n-s)=as+bn+b(n-s)<as+sn+n(n-s)=n^2$. This is a contradiction. So $s<b$.

Let (X,Y) be a bipartition of $K_{n,n}$ with $X=\{u_0,u_1,\dots,u_{n-1}\}$, $Y=Y_1\cup Y_2$, $Y_1\cap Y_2=\varnothing$, and let $Y_1=\{v_0,v_1,\dots,v_{b-1}\}$, $Y_2=\{z_0,z_1,\dots,z_{n-b-1}\}$. Orient $Y_1\to X\to Y_2$. Now, we obtain an oriented graph D (see Fig.1). The in-degree of each vertex u_i of X in D is b. The in-degree of each vertex v_i of Y_1 in D is 0 and the in-degree of each vertex z_j of Y_2 in D is n. Denote the vertex set $\{v_0,\dots,v_{b-s-1}\}\subseteq Y_1$ by Y_1'. By $s<b$, $|Y_1'|\ge 1$.

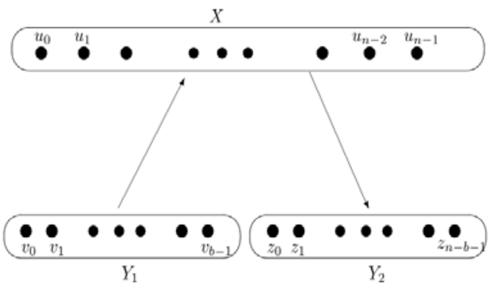

FIGURE I. THE ORIENTED GRAPH D

Algorithm 3.1.

INPUT: the above oriented graph D with three in-degrees 0, b and n.

0. Set $l_x:=0$ for every $x=0,1,\dots,b-s-1$, $r_y:=n$ for every $y=0,1,\dots,n-b-1$, $i:=0$, $j:=0$.

1. If $l_i\ge b$, $i:=i+1$.

2. If $r_j\le b$, $j:=j+1$.

3. Choose $u\in X$ satisfying $v_i\to u\to z_j$ in D. Reverse $v_i\to u\to z_j$ in D. Obtain D^*. $D:=D^*$.

4. Set $l_i:=l_i+1$ and $r_j:=r_j-1$.

5. If $i=b-s-1$ and $l_i=b$, output D. Otherwise, go to step 1.

Theorem 3.2. Let $a=0$ and $a+b<n$. Then Algorithm 3.1 outputs D which has only two in-degrees 0 and b.

Algorithm 3.2

INPUT: the above oriented graph D with four in-degrees a, b, $n-a$ and $n-b$.

0. Set $l_x:=n-a$ for every $x=0,1,\dots,s-1$, $r_y:=n-b$ for every $y=0,1,\dots,n-s-1$, $i:=0$, $j:=0$.

1. If $r_j\ge b$, $j:=j+1$.

2. If $l_i\le b$, $i:=i+1$.

3. Choose $w\in\{w_j,w_{(j-1)(\bmod (n-s))},w_{(j-2)(\bmod (n-s))},\dots,w_{(j-b+1)(\bmod (n-s))}\}$ satisfying $z_j\to w\to v_i$ in D. Reverse $z_j\to w\to v_i$ in D. Obtain D^*. $D:=D^*$.

4. Set $r_j := r_j + 1$ and $l_i := l_i - 1$.

5. If $j = n - s - 1$ and $r_j = b$, output D. Otherwise, go to step 1.

Theorem 3.3. Let $0 < a < b \le n - s$, $a \le s < t$ and $a + b < n$. Then Algorithm 3.2 outputs D which has only two in-degrees a and b.

Algorithm 3.3

INPUT: the above oriented graph D with four in-degrees a, b, $n + s - b$ and $s - a$.

0. Set $l_x := s - a$ for every $x = 0,1,\ldots,s-1$, $r_y := n + s - b$ for every $y = 0,1,\ldots,n-s-1$, $i := 0$, $j := 0$.

1. If $l_i \ge b$, $i := i + 1$.

2. If $r_j \le b$, $j := j + 1$.

3. Choose $q \in \{u_i, u_{(i-1)(\text{mod } s)}, \ldots, u_{(i-a+1)(\text{mod } s)}\} \cup (X_2 \setminus \{w_j, w_{(j-1)(\text{mod } (n-s))}, \ldots, w_{(j-b+s+1)(\text{mod } (n-s))}\})$ satisfying $v_i \to q \to z_j$ in D. Reverse $v_i \to q \to z_j$ in D. Obtain D^*. $D := D^*$.

4. Set $l_i := l_i + 1$ and $r_j := r_j - 1$.

5. If $i = s - 1$ and $l_i = b$, output D. Otherwise, go to step 1.

Theorem 3.4. Let $0 < a < b$, $n - s < b$, $s < b$, $a \le s < t$ and $a + b < n$. Then Algorithm 3.3 outputs D which has onlu two in-degrees a and b.

Algorithm 3.4.

INPUT: the above oriented graph D with five in-degrees a, b, $n - s$, s, and n.

0. Set $l_x := n - s$ for every $x = 0,1,\ldots,a-1$, $r_y := s$ for every $y = 0,1,\ldots,b-1$, $r_k := n$ for every $k = b, b+1, \ldots, n-a-1$, $i := 0$, $j := 0$.

1. If $r_j \le b$, $j := \min\{\beta : r_\beta > b, j+1 \le \beta \le n-a-1\}$.

2. If $l_i \ge b$, $i := i + 1$.

3. Choose $u \in X_1$ satisfying $v_i \to u \to z_j$ in D. Reverse $v_i \to u \to z_j$ in D. Obtain D^*. $D := D^*$.

4. Set $r_j := r_j - 1$ and $l_i := l_i + 1$.

5. If $j = n - a - 1$ and $r_j = b$, output D. Otherwise, go to step 1.

Theorem 3.5. Let $0 < a < b$, $n - s < b$, $s \ge b$, $a \le s < t$ and $a + b < n$. Then Algorithm 3.4 outputs D which has only two in-degrees a and b.

Algorithm 3.5.

INPUT: the above oriented graph D with five in-degrees a, b, $n - a$, $n + s - b - a$, and $n - b$.

0. Set $l_x := n - a$ for every $x = 0,1,\ldots,a-1$, $r_y := n + s - b - a$ for every $y = 0,1,\ldots,b-a-1$, $r_k := n - b$ for every $k = b-a, b-a+1, \ldots, n-a-1$, $i := 0$, $j := 0$.

1. If $r_j \ge b$, $j := j + 1$.

2. If $l_i \le b$, $i := i + 1$.

3. Choose $w \in \{w_j, w_{(j-1)(\text{mod } (n-a))}, w_{(j-2)(\text{mod } (n-a))}, \ldots, w_{(j-b+1)(\text{mod } (n-a))}\}$ satisfying $z_j \to w \to v_i$ in D. Reverse $z_j \to w \to v_i$ in D. Obtain D^*. $D := D^*$.

4. Set $r_j := r_j + 1$ and $l_i := l_i - 1$.

5. If $j = n - a - 1$ and $r_j = b$, output D. Otherwise, go to step 1.

Theorem 3.6. Let $0 < a < b$, $s < a$, $s < t$ and $a + b < n$. Then Algorithm 3.5 outputs D which has only two in-degrees a and b.

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